Preprint:APS

Summer Project Report

Yi Zhao

(Dated: August 8, 2019)

Abstract

In this project, we propose a non-Hermitian, time-dependent, two-level system featuring spontaneous PT symmetry breaking. Interesting features robust against shift in initial conditions are present in the dynamics of the system; for instance, the norms of the coefficients of the instantaneous eigenstates are always the same in the time period where both the eigenstates possess the PT symmetry. These features hint at ways to experimentally prepare a light field with two modes of the same amplitude. They also motivate future research on whether similar patterns can be found in other, many-body systems with spontaneous PT symmetry breaking.

I. BACKGROUND

It is well known that in the framework of Quantum Mechanics the time evolution of physical states is governed by a Hermitian Hamiltonian, whose spectrum, corresponding to the energy levels of the system, consists of real numbers only. In the last decade of the last century, however, several systems governed by non-Hermitian Hamiltonians were also found to have real spectra [1]. This founding motivated the subsequent interest in the dynamics of as well as physical meanings of non-Hermitian systems. Besides the ambitious work exploring the possibility of new kinds of Quantum Theories with non-Hermitian Hamiltonians [2], a direct application of the solution of non-Hermitian systems to wave-guide optics was stressed in the recent literature [3, 4]. In the latter case, even a negative (positive) imaginary part in the energy spectrum can carry physical meanings, for it results in an exponential growth (decay) and hence can be interpreted as gain (loss) applied to an optical system in an experiment.

To investigate the dynamics of a system governed by a non-Hermitian Hamiltonian in a simpler setting, several attempts were present to generalize the Hamiltonian appearing in the Landau-Zener model, a textbook toy model first proposed to understand the dynamics of transitions between two quantum states. As one of the discoveries, adding a non-Hermitian, time-dependent term to the Hamiltonian can quench the transition between the two states and hence allow the parameters of the system to change faster while keeping the system not far from one of its instantaneous eigenstates [5]. This happens mainly because the non-Hermitian term introduces imaginary parts in the eigenengergies of the two states, separating their energy levels further and consequently making the jump between them less likely to occur. It is the non-Hermiticity that leads to those new patterns in the system's dynamics.

Another interesting feature unique to non-Hermitian systems is the presence of exceptional points. A system is said to be at the exceptional point when the its eigenstates coalesce i.e. do not fully span the space consisting of all the possible states of the system. While mathematically the presence of those points is a straightforward result of linear algebra, their physical significance is less well understood. Given that in the parameter space of the Hamiltonian an exceptional point can be viewed as a singularity generating nontrivial topological features, the novelty the exceptional point introduces is manifest in many-body systems [6], which are rich of topological phenomena.

It is also possible to study the dynamics near the exceptional points under the context of Parity-time (PT) symmetry breaking. Conventionally, the PT reversal operator $\mathcal{PT}$, the operator that switches the sign of space and time of a system's dynamics, is defined such that [2, 7]

$$\mathcal{PT}x\mathcal{PT}^{-1} = -x, \quad \mathcal{PT}p\mathcal{PT}^{-1} = p, \quad \mathcal{PT}i\mathcal{PT}^{-1} = -i. \tag{1}$$

The last constraint is required to maintain the canonical commutator relation $[x, p] = i$. A system is hence called *PT-symmetric* (or having *PT symmetry*) if its Hamiltonian H is invariant under the action of $\mathcal{PT}$ operator i.e. $\mathcal{PT}H\mathcal{PT}^{-1} = H$. This definition also applies to the eigenstates of the system. However, as it is in the general context of symmetry breaking, the eigenstates of the system do not necessarily possess the symmetry of the system. Based on this consideration, we say that a system undergoes *PT-symmetry breaking* if its Hamiltonian has the PT symmetry while some of its eigenstates does not. As is shown by the example under investigation in our project, the occurrence of PT-symmetry breaking is directly related to exceptional points in some cases.

In this report, we propose a non-Hermitian generalization of the Landau-Zener model. We see that intriguing feature of its dynamics takes place between the exceptional points, and an analytic explanation of this feature is provided. We also examine the system under the context of PT-symmetry breaking and attempt analyze the latter one's role in the occurrence of the interesting dynamics. Our result can be also potentially applied to optics, enlightening a way to experimentally prepare a light field with some desired special properties.

II. RESULTS

We first list some general features of the system governed by the time-dependent, non-Hermitian Hamiltonian of our concern

$$H(t) = \begin{pmatrix} i\delta t & \omega_0 \\ \omega_0 & -i\delta t \end{pmatrix}, \quad \delta, \omega_0 > 0, \tag{2}$$

whose exceptional points are $\pm t_e$ for $t_e = \omega_0/\delta$. This system can be realized as a spin under an imaginary external field. Under this consideration, we decide the representation of the $\mathcal{PT}$ operator to be

$$\mathcal{PT}K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

where K is the operator performing complex conjugate. With this at hand, we are able to examine the eigenstates of the system and decide whether PT-symmetry breaking happens in the following three time periods. For simplicity of notation, from now on $|\psi_\pm(t)\rangle$ denote the eigenstates of H at time t. We also define

$$\theta(t) = \arcsin \frac{\omega_0}{\delta t} \tag{4}$$

for $t \notin (-t_e, t_e)$.

- $t \in (-\infty, -t_e)$.

$$E_\pm = \pm i\sqrt{\delta^2 t^2 - \omega_0^2}. \tag{5}$$

$$|\psi_+\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix}, \quad |\psi_-\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} \end{pmatrix}. \tag{6}$$

$$\mathcal{PT}\,|\psi_\pm(t)\rangle = -i\,|\psi_\mp(t)\rangle\,. \tag{7}$$

  The instantaneous eigenstates *break* the PT-symmetry of the system since the $\mathcal{PT}$ operator maps each eigenstate to the other one, as shown in Equ.(7).

- $t \in (-t_e, t_e)$.

$$E_\pm = \pm\sqrt{\omega_0^2 - \delta^2 t^2}, \tag{8}$$

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm\sqrt{1 - \left(\frac{\delta t}{\omega_0}\right)^2} - i\frac{\delta t}{\omega_0} \end{pmatrix}. \tag{9}$$

$$\mathcal{PT}\left|\psi_\pm(t)\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm\sqrt{1-\left(\frac{\delta t}{\omega_0}\right)^2} + i\frac{\delta t}{\omega_0} \\ 1 \end{pmatrix} \propto \left|\psi_\pm(t)\right\rangle. \tag{10}$$

Since timing a state with a constant number does not change its physical meaning, the instantaneous eigenstates preserve the $\mathcal{PT}$-symmetry of the system.

- $t \in (t_e, +\infty)$.

$$E_\pm = \pm i\sqrt{\delta^2 t^2 - \omega_0^2}, \tag{11}$$

$$\left|\psi_+\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} \end{pmatrix}. \quad \left|\psi_-\right\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -i\cos\frac{\theta}{2} \end{pmatrix}. \tag{12}$$

$$\mathcal{PT}\left|\psi_\pm(t)\right\rangle = i\left|\psi_\mp(t)\right\rangle. \tag{13}$$

Unsurprisingly, the situation here is similar to the first case. The PT-symmetry breaking occurs.

Based on above observation, we remark that the exceptional points are crucial time moments in establishing and breaking the PT symmetry.

The dynamics between the exceptional points are also interesting. Given that we have obtained a continuous definition of $\left|\psi_\pm(t)\right\rangle$, from now on we may decompose the state of the system at any time as

$$\left|\psi(t)\right\rangle = c_1(t)\left|\psi_+(t)\right\rangle + c_2(t)\left|\psi_-(t)\right\rangle. \tag{14}$$

As shown in the numerical results in Fig.1, the norms of $c_1(t)$ and $c_2(t)$ are equal to each other for any t between the exceptional points. More interestingly, this phenomenon is robust against change in initial conditions as long as the initial condition is set at a time $-T$ that is ancient enough. More peculiarities may be observed in Fig.2 where we plot the time evolution in complex planes. To name one, the traces of $c_1(t)$ and $c_2(t)$ in Fig.2b are symmetric with respect to the asymptotic axis in Fig.2a. This hints at the presence of some conserved quantities.

Mathematically, we are able to show that the difference π between the asymptotic phases of the $c_1(t)$ and $c_2(t)$ as t approaches t_e (see in Fig.2a) directly contributes to the symmetric behavior of the traces of $c_1(t)$ and $c_2(t)$ in Fig.2b. The phenomenon $|c_1(t)| = |c_2(t)|$ for $t \in (-t_e, t_e)$ is a straightforward consequence of that symmetric behavior.

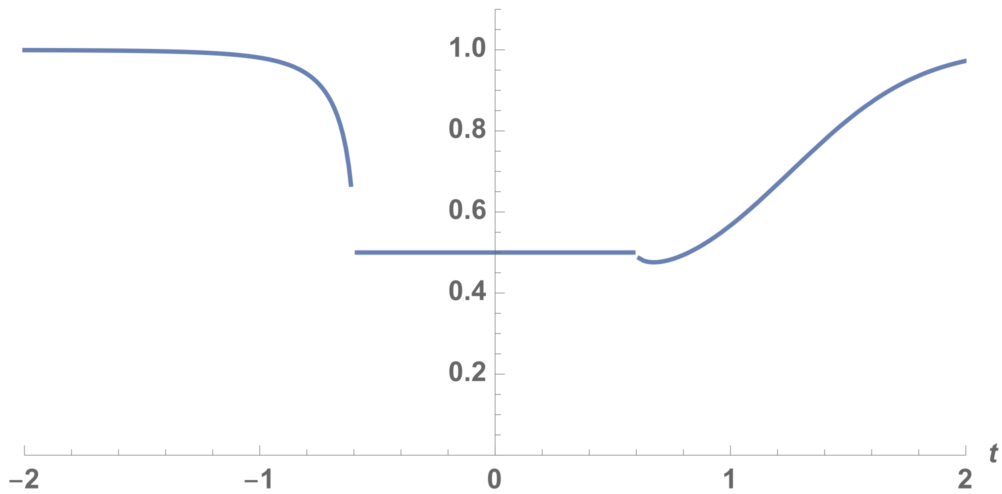
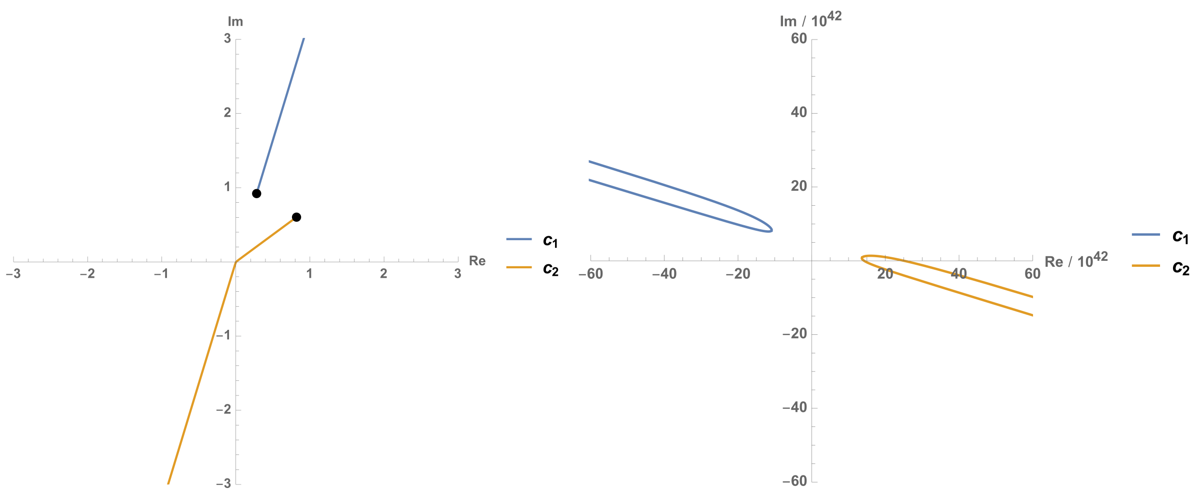


FIG. 1: Time evolution of the norm of the normalized coefficient $|c_1|^2/(|c_1|^2 + |c_2|^2)$. The seeming discontinuity near the exceptional points $t = \pm t_e$ is due to the numerical failure in performing the inverse of a singular matrix during decomposing. Nevertheless, the first-order derivative is discontinuous at those points $\pm t_e$. Initial conditions: $c_g(-T) = 0.6 + 0.8i$, $c_e(-T) = 0.3$, $T = 20$. Parameters: $\delta = 0.5$, $\omega_0 = 0.3$, and hence $t_e = 0.6$.



(a) $t \in (-T, -T + 2)$

(b) $t \in (-t_e, t_e)$

FIG. 2: The dynamics of c_1 and c_2 in the complex plane. Black dots denote the initial states, which are chosen to avoid triviality: $c_g(-T) = 0.8 + 0.6i, c_e(-T) = -\sqrt{0.91} + 0.3i$. Other parameters: $\omega_0 = 0.3, \delta = 0.5, T = 20$, and hence $t_e = 0.6$.

III. CONCLUSION

In conclusion, the system governed by the non-Hermitian, time-dependent Hamiltonian H in Equ.(2) features spontaneous PT-symmetry breaking and peculiar dynamics robust to change of initial conditions. Both of these two phenomena are closely related to the presence of exceptional points, whose existence is impossible in the usual Hermitian systems. This finding inspires further investigation into a many-body generalization of the system and also may be applied to optics. Should one be able to construct a one-dimensional optical system whose Maxwell Equation is similar to the Schrödinger Equation $i\,|\dot{\psi}(t)\rangle = H\,|\psi\rangle$ in form, it would be possible to prepare experimentally two modes of light with equal amplitudes.

Although we are able to provide a mathematical explanation to the peculiar dynamics, attributing them to the asymptotically conserved phase difference, it remains unclear whether the conserved quantity is a result of PT symmetry as is usually suggested by the Noether's theorem. Another future direction is to explain some features of the dynamics analytically to further detail. Possible goals are the jump in first-order time derivative of the quantity $|c_1|^2/(|c_1|^2 + |c_2|^2)$ at the exceptional points and the process where the phase difference is asymptoticlly formed.

IV. ACKNOWLEDGEMENTS

I would like to thank Abhinav Prem and Biao Lian for guiding me into this field and Professor Shivaji Sondhi for helping this study experience happen in the first place. I also acknowledge the financial support offered by the Office of Undergraduate Research, Princeton University.

[1] C. M. Bender and S. Boettcher, Real spectra in non-hermitian hamiltonians having $\mathcal{PT}$ symmetry, Phys. Rev. Lett. **80**, 5243 (1998).

[2] C. M. Bender, Making sense of non-Hermitian Hamiltonians, Rep. Prog. Phys. **70**, 947 (2007).

[3] B. T. Torosov and N. V. Vitanov, Pseudo-Hermitian Landau-Zener-Stückelberg-Majorana model, Phys. Rev. A **96**, 013845 (2017).

[4] R. El-Ganainy *et al.*, Non-Hermitian physics and $\mathcal{PT}$ symmetry, Nat. Phys. **14**, 11 (2018).

[5] B. T. Torosov, G. Della Valle, and S. Longhi, Non-Hermitian shortcut to adiabaticity, Phys. Rev. A **87**, 052502 (2013).

[6] T. E. Lee, Anomalous edge state in a non-Hermitian lattice, Phys. Rev. Lett **116**, 133903 (2016).

[7] M. Klett *et al.*, Relation between $\mathcal{PT}$-symmetry breaking and topologically nontrivial phases in the su-schrieffer-heeger and kitaev models, Phys. Rev. A **95**, 053626 (2017).