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Geodesics of Kerr Space-time: Equatorial and General Geodesics of Kerr Black Hole

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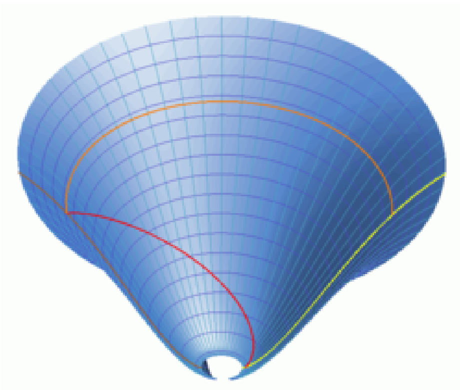

July 15, 2019

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Abstract

*This paper, exploratory in nature (**i.e. most of the work so far is a synthesis of existing research paper/books**), seeks to study the geodesics motion outside a Kerr black hole (i.e. restricted to the region outside the outer horizon). With a basic understanding of the concepts in differential geometry, namely smooth manifolds, tensors, semi-Riemannian geometry, and the basics of geodesics and Kerr metrics, this paper then discussed equatorial geodesics, the geodesics with $\theta \equiv \frac{\pi}{2}$. With detailed outlining about the properties of equatorial geodesics, the paper then discussed about more general cas, where Hamilton-Jacobi approach is applied.*

I. Historical Backgrounds

In general relativity, a geodesic is a generalization of the notion of a "straight line" to curved spacetime. Under this consideration, a freely moving or falling particle's trajectory is always considered to be a geodesics. In particular, the world line(frame) of a particle free from all external, non-gravitational force is a particular type of geodesic.

Moreover, gravity can be regarded as not a force but a consequence of a curved spacetime geometry where the source of curvature is the stress-energy tensor (representing matter, for instance). Thus, for example, the path of a planet orbiting a star is the projection of a geodesic of the curved $4D$ spacetime geometry around the star onto $3D$ space. In this paper, we'll focus on studying the properties of rotating black holes.

To mathematically analyze the properties of such black holes, we first restrict our attention to uncharged particles. To study the properties of uncharged particles, we employed a system of metric, the Kerr metric. The Kerr metric is a generalization of the Schwarzschild metric, which described the geometry of spacetime around an uncharged, spherically-symmetric, and non-rotating body.

According to the Kerr metric,rotating black-holes should exhibit frame-dragging (also known as Lense-Thirring precession), a distinctive prediction of general relativity. Roughly speaking, this effect predicts that objects coming close to a rotating mass will be entrained to participate in its rotation, not because of any applied force or torque that can be felt, but rather because of the swirling curvature of spacetime itself associated with rotating bodies. At close enough distances, all objects - even light - must rotate with the black-hole; the region where this holds is called the ergosphere.

II. Mathematical Basis

i. Tensors

The notion of tensor field on a manifold generalizes the notion of real-valued function, vector field, and one-form, and thus provides the mathematical means of describing more complicated objects on a manifold.

Formally, let $V_1, V_2, ..., V_s$ be modules over a ring K. Then $V_1 \times V_2 \times ... \times V_s$, the set of all s-tuples $(v_1, v_2, ..., v_s)$ with $v_i \in V_i$, is again a module over K. If W is also a module over K, then a function

$$A : V_1 \times V_2 \times ... \times V_s \to W$$

is *K-multilinear* provided A is K-linear in each slot. Similarly, let V^* be the set of all K-linear functions from V to K. Then V^* makes a *dual module* of V.

Definition 1. For integers $r \geq 0, s \geq 0$ not both zero, a K-multilinear function $A : (V^*)^r \times V^s \to K$ is called the tensor of type (r, s) over V.

A tensor field over a smooth manifold M is a tensor over the $\mathfrak{F}$-mutilinear function

$$A : \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s \to \mathfrak{F}(M)$$

. Hence A takes in r one-forms $\theta^1, \theta^2, ..., \theta^r$ and s vector fields $X_1, ..., X_s$ produces a real-valued function

$$f = A(\theta^1, \theta^2, ..., \theta^r, X_1, ..., X_s) \in \mathfrak{F}(M).$$

Here θ^i occupies the ith *contravariant* slot, X_j the jth *covariant slot* of A.

ii. Semi-Riemannian Manifolds

With the definition of tensor field, we are now ready to provide a generalization to the familiar geometry of the usual Euclidean space $\mathbb{R}^3$

Definition 2. A *metric tensor* $\mathbf{g}$ on a smooth manifold M is a symmetric nondegenerate (0,2)tensor field on M of constant index.

In other words, $\mathbf{g} \in \mathfrak{T}^2_0(M)$ smoothly assigns to each point p of M a scalar product $\mathbf{g}_p$ on the tangent space $T_p(M)$, and the index of $\mathbf{g}_p$ is the same for all p.

Thus a semi-Riemannian manifold is an ordered pair (M,g): two different metric tensors acting on the same manifold would constituent different semi-Riemmanian manifolds. The common value v of index $\mathbf{g}_p$ on a semi-Riemannian manofold M is called the *index* of $M : 0 \leq v \leq n = \dim M$. If $v = 0$, M is just the Riemannian manifold in the usual sense.

If $x^1, x^2, ..., x^n$ is a coordinate system on $\mathscr{U}$ the components of the metric tensor $\mathbf{g}$ on $\mathscr{U}$ are

$$g_{ij} = \mathbf{g}(\partial_i, \partial_j) \doteq \langle \partial_i, \partial_j \rangle$$

Since $\mathbf{g}$ is symmetric $g_{ij} = g_{ji}$. Finally on (u) the matric tensor can be written as

$$\mathbf{g} = \Sigma g_{ij} dx^i \otimes dx^j$$

Recall that, the basis theorem of usual Euclidean space asserts that that for each point $p \in \mathbb{R}^n$ it is possible to find a basis $\partial_1, ..., \partial_n$ in the tangent space $T_p(M)$ such that $v_p = \Sigma v^i \partial_i$ Thus the dot product on $\mathbb{R}^n$ give rise to a metric tensor on $\mathbb{R}^n$ with

$$\langle v_p, w_p \rangle = v \cdot w = \Sigma v^i w^i.$$

For an integer ν with $0 \leq v \leq n$, changing the first ν plus signs above to minus gives a metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{\nu} v^i w^i + \sum_{i=\nu+1}^{n} v^i w^i$$

of index ν. We denote the resulting *semi-Euclidian space* $\mathbb{R}^n_\nu$. It reduces to the usual Euclidean space if $\nu = 0$.

The geometric significance of the index of a semi-Riemannian manifold derives from the following trichotomy:

Definition 3. A tangent vector v to manifold M is

spacelike if $\langle v, v \rangle > 0$ or $v = 0$,
null if $\langle v, v \rangle = 0$ and $v \neq 0$,
timelike if $\langle v, v \rangle < 0$

The terminology derives from relativity theory, and particularly in the Lorentz case, null vectors are also said to be *lightlike*.

Now suppose V and W are two vector fields on semi-Riemannian manifold, we then define a new vector field on M whose value at each point $p \in M$ is the rate of change of W in the V_p direction.

Definition 4. Let $u^1, u^2, ..., u^n$ be the natural coordinates on $\mathbb{R}^n_\nu$, the vector field

$$D_V W = \Sigma V(W^i) \partial_i$$

is called the *natural coordinate derivative* of W with respect to V.

In light of this consideration, it is necessary to axiomatizing its properties so that it would be applicable to an arbitrary semi-Riemannian manifold.

Definition 5. A *connection* D on a smooth manifold M is a function $D : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

(1) $D_V W$ is $\mathfrak{F}(M)$-linear in V,

(2) $D_V W$ is $\mathbb{R}$-linear in V,

(3) $D_V(fW) = (Vf)W + fD_V W$ for $f \in \mathfrak{F}(M)$.

$D_V W$ is called the *covariant derivative* of W with respect to V for the connection D.

iii. Parallel Translations

The simplest case of a vector field on a mapping is a vector field Z on a curve $\alpha : I \to M$. Z smoothly assigns to each $t \in I$ a tangent vector to M at $\alpha(t)$. For example, the velocity α' is a vector field on α, as is the restriction V_α of any $V \in \mathfrak{X}(M)$. The set $\mathfrak{X}(\alpha)$ of all (smooth) vector fields on α is a module over $\mathfrak{F}(I)$.

When M is a semi-Riemannian manifold there is a natural way to define the vector rate of change Z' of a vector field $Z \in \mathfrak{X}(\alpha)$.

Proposition 1. *Let $\alpha : I \to M$ be a curve in a semi-Riemannian manifold M. Then there is a unique function $Z \to Z' = DZ/dt$ from $\mathfrak{X}(\alpha)$ to $\mathfrak{X}(\alpha)$, called the induced covariant derivative such that*

(1) $(aZ_1 + aZ_2)' = aZ_1' + aZ_2'$ $(a, b \in \mathbb{R})$,

(2) $(hZ)' = (dh/dt)Z + hZ'$ $(h \in \mathfrak{F}(I))$,

(3) $(V_\alpha)'(t) = D_{\alpha'(t)}(V)$ $(t \in I, V \in \mathfrak{X}(M))$.

In the special case where $Z = \alpha'$ the derivative $Z' = \alpha''$ is called the *acceleration* of the curve α. If we assume that α lines in the domain of a single coordinate system $x^1, x^2, ..., x^n$. By the basis theorem, if $Z \in \mathfrak{X}(\alpha)$, then at $\alpha(t)$,

$$Z(t) = \Sigma Z(t)x^i\partial_i = \Sigma(Zx^i)(t)\partial_i.$$

Then from (3) we have

$$Z' = \sum \frac{dZ^i}{dt}\partial_i + \sum Z^i D_{\alpha'}(\partial_i)$$

If we write out the derivative explicitly, the equation becomes

$$Z' = \sum_k \{\frac{dZ^k}{dt} + \sum_{i,j} \Gamma^k_{i,j} \frac{d(x^i \circ \alpha)}{dt} Z^j\}\partial_k$$

Where $\Gamma^k_{i,j}$ is defined as

$$\Gamma^k_{i,j} = \frac{1}{2}\sum_m g^{km}\{\frac{\partial \mathbf{g}_{jm}}{\partial x^i} + \frac{\partial \mathbf{g}_{im}}{\partial x^j} + \frac{\partial \mathbf{g}_{ij}}{\partial x^m}\}.$$

We donate this symbol as **Christoffel symbol.**

If $Z' = 0$, then Z is said to be *parallel*. This formula shows that the equation $Z' = 0$ is equivalent to a system of linear ordinart differential equations. Thus the fudamental existence and uniqueness theorem of such system yields:

Proposition 2. *For a curve $\alpha : I \to M$, let $a \in I$ and $z \in T_{\alpha(a)}(M)$. Then there is a unique parallel vector field Z on α such that $Z(a) = z$.*

In the notation of the proposition, if $b \in I$ then the function

$$P = P^b_a(\alpha) : T_p(M) \to T_q(M)$$

sending each z to $Z(b)$ is called *parallel translation along α* from $p = \alpha(a)$ to $q = \alpha(b)$.

Lemma II.1. *Parallel translation is a linear isometry.*

Proof. With the notation as above, let $v, w \in T_p(M)$ correspond as in the proposition to parallel vector fields V, W. Since $V + W$ is also parallel, $P(v + w) = (V + W)(b) = V(b) + W(b) = P(v) + P(w)$. Similarly, $P(cv) = cP(v)$. Thus P is linear.
If $P(v) = 0$ then by the uniqueness in the proposition, V can only be the identically zero vector field on α. Hence $v = V(\alpha) = 0$. Thus P is one-to-one, and since tangent spaces to M have the same dimension, P is linear isomorphism.
Finally, for V, W as above,

$$\frac{d}{dt}\langle V, W\rangle = \langle V', W\rangle + \langle V, W'\rangle = 0$$

Hence $\langle V, W\rangle$ is constant, so

$$\langle P(v), P(w)\rangle = \langle V(b), W(b)\rangle = \langle V(a), W(a)\rangle = \langle a, b\rangle$$

□

iv. Geodesics

With all the mathematical basics, we now generalize the Euclidean notion of straight line. A *geodesic* in a semi-Riemannian manifod M is a curve $\gamma : I \to M$ whose vector field γ' is parallel. Equivalently, geodesics are the curves of acceleration zero $\gamma'' = 0$.

Corollary II.1. *Let $x^1, ..., x^n$ be a coordinate system on $\subset M$. A curve γ in $\mathscr{U}$ is a geodesic of M if and only if its coordinate functions $x^k \circ \gamma$ satisfy*

$$\frac{d^2(x^k \circ \gamma)}{dt^2} + \sum i, j\Gamma^k_{ij}(\gamma)\frac{d(x^i \circ \gamma)}{dt}\frac{d(x^j \circ \gamma)}{dt} = 0$$

for $1 \leq k \leq n$.

In fact, these expressions are the components of γ'' relative to the coordinate vector fields $\partial_1, ..., \partial_n$. As discussed above, the existence and uniqueness theorem for ordinary differential equations gives the following local result.

Lemma 1. *If $v \in T_p(M)$ there exists an interval I about 0 and a unique geodesic $\gamma : I \to M$ such that $\gamma'(0) = v$.*

Lemma 2. *Let $\alpha, \beta : I \to M$ be geodesics. If there is a number $a \in I$ such that $\alpha'(a) = \beta"(a)$, then $\alpha = \beta$.*

Example II.1 (**Geodesics of Semi-Euclidean Space.**). *For natural coordinates the Christoffel symbols vanish, so the geodesic equations become*

$$\frac{d^2(u^i \circ \gamma)}{dt^2} = 0(1 \leq i \leq n).$$

Thus $u^i(\gamma(t)) = p^i + tv^i$ for all t, where p^i and v^i are arbitrary constants.

III. Conclusion

The discussion so far have shown that the geodesics is a powerful tool that provides a generalization to the Euclidean straight line. This proves to be a powerful tool for us to analyze the properties in general relativity, where the trajectory a particle is moving can be best modeled with geodesics.

IV. Upcoming work

In the first half of the research, the various basics of differential geometry is being explored. The report presented a wide range of materials in differential geometry that is essential for the understanding of the works in Kerr Metrics. The materials covered including smooth manifolds, tensors, semi-Riemannian manifolds, parallel translations, geodesics and Kerr metrics.

With an understanding of all the essential materials covered in the Kerr Black metrics, the upcoming work is likely to have a heavier focus on discussing the key materials central to the bound in non-rotating black hole. As outlined in the research outline, an approach moving from the specific case of geodesics of Kerr Black hole, namely the equatorial geodesics to a more general case, namely the geodesics with varying angle, would be adapted. In light of all the previous research, it is hoped that an better approximation on the current bond could be obtained by the end of this research.

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